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Closed loop solitons and sigma functions: classical and quantized elasticas with genera one and two

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Abstract

Closed loop solitons in a plane, whose curvatures obey the modified Korteweg–de Vries equation, were investigated. It was shown that their tangential vectors are expressed by ratio of Weierstrass sigma functions for genus one case and ratio of Baker's sigma functions for the genus two case. This study is closely related to classical and quantized elastica problems. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

This article is on loop solitons of genera one and two. We give them as ratios of sigma functions; for the case of genus one, the sigma functions are Weierstrass sigma functions while for the case of genus two, they are Baker's hyperelliptic sigma functions.

First, we mention background of the study, which is related to elastica problem. Elastica problem was proposed by James Bernoulli at the end of 17th century and Euler and his nephew Daniel Bernoulli investigated the problem [9,19,20]. The elastica problem is to determine all allowed shapes of a thin elastic beam in a plane. The allowed shape is realized as a curve with the local minimum of the energy

$$E = \int \mathrm{d}s\, k^2, \tag{1.1}$$

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where s is its arclength and k its curvature. For its tangential angle ϕ and $g := \exp(\sqrt{-1}\phi)$, the integrand is given by

$$k^2\,\mathrm{d}s = -(g^{-1}\,\mathrm{d}g)(*g^{-1}\,\mathrm{d}g) = (\partial_s\phi)^2\,\mathrm{d}s, \tag{1.2}$$

where $*$ is the Hodge star, a map from one-form to zero-form, and $\partial_s := \mathrm{d}/\mathrm{d}s$. Thus the elastica problem is the oldest harmonic map (sigma-model) problem from a circle S^1 or a real line $\mathbb{R}$ to $U(1)$. This problem was essentially solved by Euler using numerical computations [9,19].

For a given elastica in two-dimensional Euclidean space, it is characterized by the affine coordinate $(X^1(s), X^2(s))$ or $Z(s) = X^1(s) + \sqrt{-1}X^2(s)$. Then its curvature k is given by $k := (1/\sqrt{-1})\partial_s \log \partial_s Z$. The shape locally minimizing the energy (1.1) is given by the differential equation [11,12]:

$$C\partial_s k + \tfrac{3}{2}k^2\partial_s k + \partial_s^3 k = 0. \tag{1.3}$$

This is known as the static modified Korteweg–de Vries (SMKdV) equation.

As I showed in Ref. [12], when we deal with a non-stretching closed elastica in thermal bath or investigate the partition function of temperature $1/\beta$,

$$\mathcal{Z} = \int \mathcal{D}Z \exp\left(-\beta \int \mathrm{d}s\, k^2\right), \tag{1.4}$$

we also need information of curves which have the excited energy. We will call such curves "quantized elastica" following [14]. This problem is associated with large polymer problem like DNA in biology and polymer physics. As shown in [12–14], the quantized elastica is classified by the modified Korteweg–de Vries (MKdV) equation,

$$C\partial_t k + \tfrac{3}{2}k^2\partial_s k + \partial_s^3 k = 0, \tag{1.5}$$

where $\partial_s := \partial/\partial s$, $\partial_t := \partial/\partial t$ and t is a deformation parameter. The orbits obeying the MKdV equation preserve local length and their first integral (1.1), i.e., $[\partial_s, \partial_t] = 0$ and $\partial_t E = 0$. Since the MKdV equation is an initial value problem, for a given smooth curve, there exists, at least, an MKdV orbit including the curve; we can classify quantized elastica using the orbits. Hence the partition function (1.4) is reduced to an integration,

$$\mathcal{Z} = \int \mathrm{d}\Omega(E) \exp(-\beta E), \tag{1.6}$$

where $\mathrm{d}\Omega(E)$ is the density of states. The evaluation of $\mathrm{d}\Omega(E)$ means the determination of the orbits of the MKdV equation whose first integral is E. In other words, in order to evaluate (1.5), we must explicitly construct the periodic solutions of MKdV equation, which are expressed by hyperelliptic functions. The hyperelliptic function is characterized by genus. As mentioned in [12], orbits with small genus is expected to largely contribute to the partition functions (1.4). Thus as an attempt, we will investigate the classical and quantized elastica of genera one and two, respectively.

Since the curve whose curvature obeys the MKdV equation was called loop soliton [7,8], our investigation is, just, a study of closed loop solitons. Here we will also note that the time development of classical elastica does not obey the MKdV equation except genus one solution case [11]. (From physical point of view, investigation of a loop soliton with higher genus is somewhat nonsense in pure kinematic study.)

In this paper, we will show that the tangential vectors of closed loop solitons with genera one and two are expressed by ratio of Weierstrass sigma functions in Section 3 and ratio of Baker's sigma functions in Section 4, respectively. This result is very interesting as we will discuss in Section 5. The sigma function, which is a well-tuned theta function, plays very important role in the studies of elliptic and hyperelliptic functions [1–4,6,15–18].

Further, our result has an effect on historical interpretation of history of science. According to Truesdell [19] and Love [9], Euler did illustrate the shape of loop soliton with genus one, our result means that Euler essentially found the elliptic and sigma functions even though he dealt only with elliptic integrations. In fact, the tangential vector $\partial_s Z(s)$ is periodic function of s. I believe that Euler noticed its properties and thus after he read Fagnano's article, his mind felt the addition formula [20].

The construction of closed loop soliton with genus two is based upon Baker's hyperelliptic function theory [1–3]. Recently Baker's approach is re-evaluated from various viewpoints [4,6,15–18]. Buchstaber et al. have been actively studying the Baker's theory from soliton theory and developing it. Grant and Ônishi found interesting relations in hyperelliptic function in terms of Baker's theory from the point of view of the number theory. As on Baker's theory of the hyperelliptic function, we start from a hyperelliptic curve and find an explicit function form without ambiguous parameter, it is very effective.

2. Basic properties of loop soliton

Let us consider a smooth immersion of a circle S^1 into the two-dimensional Euclidean space $\mathbb{E}^2 \approx \mathbb{C}$. The immersed curve C is characterized by the affine coordinate $(X^1(s), X^2(s))$. Here s is a parameter of S^1 and is, now, chosen as the arclength so that $\mathrm{d}s^2 = (\mathrm{d}X^1)^2 + (\mathrm{d}X^2)^2$. We will also use the complex expression

$$Z(s) := X^1(s) + \sqrt{-1}X^2. \tag{2.1}$$

Then $|\partial_s Z(s)| = 1$ and the curvature of C is given by

$$k(s) = \frac{1}{\sqrt{-1}}\partial_s \log \partial_s Z(s). \tag{2.2}$$

Definition 2.1. A one parameter family of curves $\{C_t\}$ for real parameter $t \in \mathbb{R}$ is called a loop soliton, if its curvature obeys the MKdV equation; for $q := \frac{1}{2}k$,

$$\partial_t q + 6q^2\partial_s q + \partial_s^3 q = 0. \tag{2.3}$$

Here we will describe the Miura map for later convenience.

Proposition 2.2 (Drazin and Johnson [5]). *For the solutions $p_\pm$ of the KdV equation*

$$\partial_t p_\pm + 6p_\pm\partial_s p_\pm + \partial_s^3 p_\pm = 0 \tag{2.4}$$

if we find a quantity q satisfying $p_{\pm} \equiv q^2 \pm \partial_s q$ over C_t and for all t, q obeys the MKdV equation (2.3).

3. Loop soliton with genus one and Weierstrass sigma function

In this section, we will deal with a loop soliton with genus one. In other words, we consider the case of $t = s/C$, i.e., the SMKdV equation

$$C\partial_s q + 6q^2\partial_s q + \partial_s^3 q = 0. \tag{3.1}$$

First, we will set up the Weierstrass $\wp$- and σ-functions.

Definition 3.1 (Wittaker and Watson [21]). For an elliptic curve

$$y^2 = 4x^3 - g_2 x + g_3 = 4(x - e_1)(x - e_2)(x - e_3). \tag{3.2}$$

1. We will define integrals u, ω_a, η_a ($a = 1, 2, 3$):

$$u := \int_{(0,0)}^{(x,y)} \frac{\mathrm{d}x}{y}, \qquad \omega_a = \int_{(0,0)}^{(e_a,0)} \frac{\mathrm{d}x}{y}, \qquad \eta_a = \int_{(0,0)}^{(e_a,0)} \frac{x\,\mathrm{d}x}{y}. \tag{3.3}$$

2. The elliptic theta function of $\tau := \omega_3/\omega_1$ is defined by

$$\theta_1(z) := \sum_{n\in\mathbb{Z}} \exp\left(2\sqrt{-1}\pi\left[\frac{1}{2}\tau\left(n-\frac{1}{2}\right)^2 + \left(n-\frac{1}{2}\right)\left(z-\frac{1}{2}\right) + \frac{1}{4}\right]\right). \tag{3.4}$$

3. The Weierstrass σ function is defined by

$$\sigma(u) := \exp\left(\frac{\eta_1 u^2}{2\omega_1}\right)\frac{\theta_1(u/2\omega_1)}{\partial_u\theta_1(u/2\omega_1)|_{u=0}}. \tag{3.5}$$

4. We also introduce the other σ-functions

$$\sigma_a(u) := \exp(-\eta_a u)\frac{\sigma(u+\omega_a)}{\sigma(\omega_a)}, \quad a = 1, 2, 3. \tag{3.6}$$

5. The Weierstrass $\wp$-function is defined by

$$\wp := -\partial_u^2 \log\sigma(u). \tag{3.7}$$

Theorem 3.2. *The shape of loop soliton with genus one is given by*

$$Z(s) = \int^s \mathrm{d}s \left(\frac{\sigma_3(s - \frac{1}{2}\omega_3 + \delta)}{\sigma(s - \frac{1}{2}\omega_3 + \delta)}\right)^2, \tag{3.8}$$

where δ is a constant parameter.

Later part of this section devotes the proof of this theorem, but first we will summarize the properties of Weierstrass (elliptic) $\wp$- and σ-functions. Let "′" be derivative in u.

Proposition 3.3 (Wittaker and Watson [21]).

1. $$\wp'(u)^2 = 4\wp^3 + g_3\wp + g_4 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3), \tag{3.9}$$
where $\wp(\omega_a) = e_a$ *and* $e_1 + e_2 + e_3 = 0$.
2. $$12\wp(u)\partial_u\wp(u) + \partial_u^3\wp(u) = 0. \tag{3.10}$$

Further, we will note the addition relations.

Proposition 3.4 (Wittaker and Watson [21]).

1. $$\wp(z) - \wp(u) = \frac{\sigma(z+u)\sigma(z-u)}{[\sigma(z)\sigma(u)]^2}, \tag{3.11}$$
2. $$\wp(u+z) - \wp(u) = -\frac{1}{2}\partial_u \frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}, \tag{3.12}$$
3. $$\wp(z+u) + \wp(z) + \wp(u) = \frac{1}{4}\left(\frac{\wp'(u) - \wp'(z)}{\wp(u) - \wp(z)}\right)^2, \tag{3.13}$$
4. $$\wp(u) - e_a = \left(\frac{\sigma_a(u)}{\sigma(u)}\right)^2. \tag{3.14}$$

Here (3.12) and (3.14) is directly derived from (3.11). (3.13) is slightly difficult but using Proposition 3.3, it can be proved.

From the definition of loop soliton, Theorem 3.2 is reduced to the following lemma.

Lemma 3.5. *For* $\rho := \wp(u) - e_3$, $\mu(u) := 1/(2\sqrt{-1})\partial_u \log\rho$ *obeys the MKdV equation*

$$6e_3\partial_u\mu + 6\mu^2\partial_u\mu + \partial_u^3\mu = 0. \tag{3.15}$$

Proof. From Proposition 3.4:

$$\wp(u+\omega_3) - \wp(u) = \frac{1}{\sqrt{-1}}\partial_u\mu, \tag{3.16}$$

$$\wp(u+\omega_3) + \wp(u) + e_3 = -\mu^2. \tag{3.17}$$

Thus for $v := u - \frac{1}{2}\omega_3$, we have the relation

$$-2\wp(v \pm \tfrac{1}{2}\omega_3) - e_3 = \mu^2(v) \pm \sqrt{-1}\partial_v\mu(v).$$

From (3.10), $p = -2\wp(u) - e_3$ satisfies the equation

$$6e_3\partial_u p + 6p\partial_u p + \partial_u^3 p = 0. \tag{3.18}$$

Using Proposition 2.2, this lemma is proved. □

Hence Proposition 3.2 is also satisfied.

Remark 3.6. We note that (3.8) is a necessary condition. If one tunes δ in (3.8) and the g_2 and g_3 in (3.2) to find a real line s in the complex plane $\{u\} = \mathbb{C}$, so that it satisfies the reality condition $|\partial_s Z| = 1$ and the closed condition $Z(s) = Z(s + 2\omega_1)$, (3.8) becomes the closed loop soliton.

4. Loop soliton with genus two and Baker's sigma function

In this section, we will deal with a hyperelliptic solution of the loop soliton whose genus is two. We will redefine the quantities $\sigma, \wp, \rho, \omega, \eta, \tau$ in the previous section. These quantities are altered to genus two versions of corresponding quantities in the previous section.

Definition 4.1 ([1,2], [4, Chapter 2], [16, pp. 384–386], [17]). For a hyperelliptic curve of genus two

$$y^2 = f(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_5 x^5, \tag{4.1}$$

where $\lambda_5 \equiv 1$ and λ_j's are complex numbers, we will give definitions as follows:

1. We choose a_j, c_j and c $(j = 1, 2)$ for

$$f(x) = P(x)Q(x),$$
$$P(x) = (x - a_1)(x - a_2), \qquad Q(x) = (x - c_1)(x - c_2)(x - c), \tag{4.2}$$

   so that curves (a_j, c_j) or (c, ∞) does not intersect each other and $\mathrm{Re}(a_j) \leq \mathrm{Re}(c_j)$, $\mathrm{Re}(a_1) \leq \mathrm{Re}(a_2)$, $\mathrm{Re}(c_5) \leq \mathrm{Re}(c)$.
2. Let us denote the homology of the hyperelliptic curve by

$$\mathrm{H}_1(X_g, \mathbb{Z}) = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\beta_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\beta_2, \tag{4.3}$$

   where these intersections are given as $[\alpha_i, \alpha_j] = 0$, $[\beta_i, \beta_j] = 0$ and $[\alpha_i, \beta_j] = \delta_{i,j}$.
3. The unnormalized differentials of first kind are defined by

$$\mathrm{d}u_1 := \frac{\mathrm{d}x}{2y}, \qquad \mathrm{d}u_2 := \frac{x\,\mathrm{d}x}{2y}. \tag{4.4}$$

4. The unnormalized differentials of second kind are defined by

$$\mathrm{d}r_1 := \frac{1}{2y}(\lambda_3 x + 2\lambda_4 x^2 + 3\lambda_5 x^3)\,\mathrm{d}x, \qquad \mathrm{d}r_2 := \frac{1}{2y}\lambda_5 x^2\,\mathrm{d}x. \tag{4.5}$$

5. The unnormalized period matrices are defined by

$$2\boldsymbol{\omega}' := \begin{bmatrix} \int_{\alpha_1}\mathrm{d}u_1 & \int_{\alpha_2}\mathrm{d}u_1 \\ \int_{\alpha_1}\mathrm{d}u_2 & \int_{\alpha_2}\mathrm{d}u_2 \end{bmatrix}, \qquad 2\boldsymbol{\omega}'' := \begin{bmatrix} \int_{\beta_1}\mathrm{d}u_1 & \int_{\beta_2}\mathrm{d}u_1 \\ \int_{\beta_1}\mathrm{d}u_2 & \int_{\beta_2}\mathrm{d}u_2 \end{bmatrix},$$
$$\boldsymbol{\omega} := \begin{bmatrix} \boldsymbol{\omega}' \\ \boldsymbol{\omega}'' \end{bmatrix}. \tag{4.6}$$

6. The normalized period matrices are given by

$$\boldsymbol{\tau} := \boldsymbol{\omega}'^{-1}\boldsymbol{\omega}'', \qquad \hat{\boldsymbol{\omega}} := \begin{bmatrix} 1_2 \\ \boldsymbol{\tau} \end{bmatrix}. \tag{4.7}$$

7. The complete hyperelliptic integral of the second kind is given as

$$2\boldsymbol{\eta}' := \begin{bmatrix} \int_{\alpha_1} \mathrm{d}r_1 & \int_{\alpha_2} \mathrm{d}r_1 \\ \int_{\alpha_1} \mathrm{d}r_2 & \int_{\alpha_2} \mathrm{d}r_2 \end{bmatrix}, \qquad 2\boldsymbol{\eta}'' := \begin{bmatrix} \int_{\beta_1} \mathrm{d}r_1 & \int_{\beta_2} \mathrm{d}r_1 \\ \int_{\beta_1} \mathrm{d}r_2 & \int_{\beta_2} \mathrm{d}r_2 \end{bmatrix}. \tag{4.8}$$

8. We will defined the Riemann theta function over $\mathbb{C}^2$ characterized by $\hat{\boldsymbol{\Lambda}}$

$$\begin{aligned} \theta \begin{bmatrix} a \\ b \end{bmatrix} (z) &:= \theta \begin{bmatrix} a \\ b \end{bmatrix} (z; \boldsymbol{\tau}) \\ &:= \sum_{n \in \mathbb{Z}^2} \exp \left[2\pi\sqrt{-1} \left\{ \frac{1}{2} {}^t(n+a)\boldsymbol{\tau}(n+a) + {}^t(n+a)(z+b) \right\} \right] \end{aligned} \tag{4.9}$$

for two-dimensional vectors a and b.

9. We will introduce a constant parameter [2, p. 343]

$$\lambda_r = \frac{(-1)^r \sqrt{P'(a_r)}}{(\sqrt{-1 f'(a_r)/4})^{1/2}}, \quad r = 1, 2. \tag{4.10}$$

We will note that these contours in the integral are, for example, given in p. 3.83 in [10]. Thus above values can be, in principle, computed in terms of numerical method for a given $y^2 = f(x)$.

Proposition 4.2 ([1], [2, pp. 316–335], [4, p. 25], [10, pp. 3.80–3.84], [17]). *The Riemannian constant $K \in \mathbb{C}^2$, $\omega_a \in \mathbb{C}^2$ are given by*

$$K = \boldsymbol{\omega}'^{-1} \left(\int_{\infty}^{(a_1,0)} \mathrm{d}\mathbf{u} + \int_{\infty}^{(a_2,0)} \mathrm{d}\mathbf{u} \right) = \delta' + \delta''\boldsymbol{\tau}, \tag{4.11}$$

$$\omega_1 := \int_{\infty}^{(a_1,0)} \mathrm{d}\mathbf{u} = \boldsymbol{\omega}'\delta_1' + \boldsymbol{\omega}''\delta_1'', \qquad \omega_2 := \int_{\infty}^{(a_2,0)} \mathrm{d}\mathbf{u} = \boldsymbol{\omega}'\delta_2' + \boldsymbol{\omega}''\delta_2'', \tag{4.12}$$

where

$$\begin{aligned} &\delta_1' := \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \qquad \delta_2' := \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \qquad \delta' = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}, \\ &\delta_1'' := \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \qquad \delta_2'' := \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \qquad \delta'' := \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}. \end{aligned} \tag{4.13}$$

Definition 4.3 ([1, p. 286], [2, pp. 336, 353, 370], [4, pp. 32, 35], [16, pp. 386–387], [17]). We will define the coordinate (u_1, u_2) in $\mathbb{C}^2$ for points (x_1, y_1) and (x_2, y_2) of the curve $y^2 = f(x)$

$$u_j := \int_{\infty}^{(x_1,y_1)} \mathrm{d}u_j + \int_{\infty}^{(x_2,y_2)} \mathrm{d}u_j. \tag{4.14}$$

1. Using the coordinate u_j, sigma functions, which are homomorphic functions over $\mathbb{C}^2$, are defined by

$$\sigma(u) = \sigma(u; \boldsymbol{\omega}) := \gamma \exp\left(-\frac{1}{2}{}^t u \boldsymbol{\eta}' \boldsymbol{\omega}'^{-1} u\right) \vartheta \begin{bmatrix} \delta'' \\ \delta' \end{bmatrix} ((2\boldsymbol{\omega}')^{-1} u; \mathbb{T}), \tag{4.15}$$

$$\sigma_r(u) = \sigma_r(u; \boldsymbol{\omega}) := \lambda_r \exp\left(-\frac{1}{2}{}^t u \boldsymbol{\eta}' \boldsymbol{\omega}'^{-1} u\right) \vartheta \begin{bmatrix} \delta_r'' \\ \delta_r' \end{bmatrix} ((2\boldsymbol{\omega}')^{-1} u; \mathbb{T}) \quad \text{for } r = 0, 1, 2, \tag{4.16}$$

where γ is a fixed constant, $\delta_0' \equiv \delta_0'' := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\lambda_0 = 1$.

2. In terms of σ-function, $\wp$-functions over the hyperelliptic curve are given by

$$\wp_{\mu\nu}(u) = -\frac{\partial^2}{\partial u_\mu \partial u_\nu} \log \sigma(u), \qquad \wp_{\mu\nu}^{(r)}(u) = -\frac{\partial^2}{\partial u_\mu \partial u_\nu} \log \sigma_r(u). \tag{4.17}$$

Theorem 4.4. *The shape of loop soliton with genus two is given by*

$$Z(s,t) = \int^s \mathrm{d}s \left(\frac{\sigma_2(u(s,t) + \omega_1 + \frac{1}{2}\omega_2 + \delta)}{\sigma_0(u(s,t) + \omega_1 + \frac{1}{2}\omega_2 + \delta)}\right)^2 \tag{4.18}$$

for $a_2 = -\frac{1}{3}\lambda_4$, *where* δ *is a constant parameter and* $u(s,t) := \begin{pmatrix} -4t \\ s \end{pmatrix}$.

Before proving this theorem, we will also review the properties of Baker's $\wp$- and σ-functions.

Proposition 4.5 ([3, pp. 155–156], [4, pp. 56–58], [6, pp. 103–104, 116], [16, p. 388], [17]). *Let us express* $\wp_{\mu\nu\rho} := \partial\wp_{\mu\nu}(u)/\partial u_\rho$ *and* $\wp_{\mu\nu\rho\lambda} := \partial^2\wp_{\mu\nu}(u)/\partial u_\mu \partial u_\nu$. *Then hyperelliptic* $\wp$*-functions obey the relations*

$$(\mathrm{H1}): \quad \wp_{2222} - 6\wp_{22}^2 = 2\lambda_3\lambda_5 + 4\lambda_4\wp_{22} + 4\lambda_5\wp_{21},$$

$$(\mathrm{H2}): \quad \wp_{2221} - 6\wp_{22}\wp_{21} = 4\lambda_4\wp_{21} - 2\lambda_5\wp_{11},$$

$$(\mathrm{H3}): \quad \wp_{2211} - 4\wp_{21}^2 - 2\wp_{22}\wp_{11} = 2\lambda_3\wp_{21},$$

$$(\mathrm{H4}): \quad \wp_{2111} - 6\wp_{21}\wp_{11} = -4\lambda_0\lambda_5 - 2\lambda_1\wp_{22} + 4\lambda_2\wp_{21},$$

$$(\mathrm{H5}): \quad \wp_{1111} - 6\wp_{11}^2 = -8\lambda_0\lambda_4 + 2\lambda_1\lambda_3 - 12\lambda_0\wp_{22} + 4\lambda_1\wp_{21} + 4\lambda_2\wp_{11}.$$

$$(\mathrm{I1}): \quad \wp_{222}^2 = 4(\wp_{22}^3 + \wp_{12}\wp_{22} + \lambda_4\wp_{22}^2 + \wp_{11} + \lambda_3\wp_{22} + \lambda_2),$$

$$(\mathrm{I2}): \quad \wp_{222}\wp_{221} = 4(\wp_{12}\wp_{22}^2 - \tfrac{1}{2}(\wp_{11}\wp_{22} - \wp_{12}^2 + \lambda_3\wp_{12} - \lambda_1) + \lambda_3\wp_{12} + \lambda_4\wp_{12}\wp_{22}),$$

$$\begin{aligned}\text{(I3)}:\quad \wp_{221}^2 &= 4(\wp_{11}\wp_{22}^2 - (\wp_{11}\wp_{22} - \wp_{12}^2 + \lambda_3\wp_{12} - \lambda_1)\wp_{22} - \wp_{11}\wp_{12}\\ &\quad + \lambda_4\wp_{11}\wp_{22} + \lambda_3\wp_{12}\wp_{22} - \lambda_4(\wp_{11}\wp_{22} - \wp_{12}^2 + \lambda_3\wp_{12} - \lambda_1)\\ &\quad + \lambda_4\lambda_3\wp_{12} - \lambda_1\wp_{22} - \lambda_1\lambda_4 + \lambda_0).\end{aligned}$$

We also have the additional formulae for them.

Proposition 4.6 (Baker [2, pp. 372, 381] and Buchstaber et al. [4, p. 112]).

1. $$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp_{22}(u)\wp_{21}(v) - \wp_{21}(u)\wp_{22}(v) - \wp_{11}(u) + \wp_{11}(v). \quad (4.19)$$
2. $$\left(\frac{\sigma_2(u)}{\sigma_0(u)}\right)^2 = \frac{1}{Q(a_r)}(\wp_{21}^{(0)}(u) + \wp_{22}^{(0)}(u)a_2 - a_2^2). \quad (4.20)$$

We note that (4.19) corresponds to (3.11), and (4.20) is a genus two version of (3.14). The similar formulae of (3.13) and (3.14) are studied in [6, pp. 114–116], but they are not so effective for this purpose. For special value (theta characteristics), we give genus two versions (4.27) and (4.28) of (3.16) and (3.17) in proof of Lemma 3.5.

Proposition 4.7.

$$\wp_{\mu\nu}^{(0)}(u) = \wp_{\mu\nu}(u - \omega_1 - \omega_2), \qquad \wp_{\mu\nu}^{(1)}(u) = \wp_{\mu\nu}(u - \omega_2),$$
$$\wp_{\mu\nu}^{(2)}(u) = \wp_{\mu\nu}(u - \omega_1). \quad (4.21)$$

Proof. Using $\theta(z) := \vartheta\begin{bmatrix}\delta_0''\\ \delta_0'\end{bmatrix}(z)$, the σ-functions are explicitly written by

$$\begin{aligned}\sigma(u) &= \gamma\exp(-\tfrac{1}{2}{}^t u\boldsymbol{\eta}'\boldsymbol{\omega}'^{-1}u + 2\pi\sqrt{-1}[\tfrac{1}{2}{}^t\delta''\boldsymbol{\tau}\delta'' + {}^t\delta''((2\boldsymbol{\omega}')^{-1}u\\ &\quad + \delta')])\vartheta((2\boldsymbol{\omega}')^{-1}u + \delta' + \boldsymbol{\tau}\delta''),\end{aligned}$$

$$\begin{aligned}\sigma_r(u) &= \lambda_r\exp(-\tfrac{1}{2}{}^t u\boldsymbol{\eta}'\boldsymbol{\omega}'^{-1}u + 2\pi\sqrt{-1}[\tfrac{1}{2}{}^t\delta_r''\boldsymbol{\tau}\delta_r'' + {}^t\delta_r''((2\boldsymbol{\omega}')^{-1}u\\ &\quad + \delta_r')])\vartheta((2\boldsymbol{\omega}')^{-1}u + \delta_r' + \boldsymbol{\tau}\delta_r''), \quad r = 0, 1, 2. \quad (4.22)\end{aligned}$$

Thus

$$\log\left(\frac{\sigma_r(u)}{\sigma(u)}\right) - \log\left(\frac{\vartheta((2\boldsymbol{\omega}')^{-1}u + \delta_r' + \boldsymbol{\tau}\delta_r'')}{\vartheta((2\boldsymbol{\omega}')^{-1}u + \delta' + \boldsymbol{\tau}\delta'')}\right)$$

is a first order polynomial of u. Thus the derivative in the definitions of $\wp$ leave only the difference of phases. Noting Proposition 4.2 and $K = (2\boldsymbol{\omega}')^{-1}(\omega_1 + \omega_2)$, we obtain the identities:

$$u = u - \omega_1 - \omega_2 + 2\boldsymbol{\omega}'K, \qquad u + 2\boldsymbol{\omega}'(\delta_1' + \boldsymbol{\tau}\delta_1'') = u - \omega_2 + 2\boldsymbol{\omega}'K,$$
$$u + 2\boldsymbol{\omega}'(\delta_2' + \boldsymbol{\tau}\delta_2'') = u - \omega_1 + 2\boldsymbol{\omega}'K. \quad (4.23)$$

Hence (4.21) is proved. □

From Definition 2.1, Theorem 4.4 is reduced to the following lemma.

Lemma 4.8. *For* $a_2 = -\frac{1}{3}\lambda_4$, $\mu(u) := \frac{1}{\sqrt{-1}}\partial_{u_2}\log(\sigma_2(u)/\sigma_0(u))$ *obeys the MKdV equation*

$$-4\partial_{u_1}\mu + 6\mu^2\partial_{u_2}\mu + \partial_{u_2}^3\mu = 0. \tag{4.24}$$

Proof. By noting (4.20) and introducing

$$\rho(u) := \wp_{21}^{(0)}(u) + \wp_{22}^{(0)}(u)a_2 - a_2^2, \tag{4.25}$$

$\mu(s)$ is given by

$$\mu(u) = \frac{1}{2\sqrt{-1}}\frac{\partial_{u_2}\rho(u)}{\rho(u)}. \tag{4.26}$$

By taking the logarithm and derivative in u_2, the addition formula (4.20) is reduced to

$$\wp_{22}(u+\omega_2) - \wp_{22}(u) = \sqrt{-1}\partial_{u_2}\mu(u+\omega_1+\omega_2). \tag{4.27}$$

Thus we must prove the relation

$$\wp_{22}(u+\omega_2) + \wp_{22}(u) + \lambda_4 + a_2 = -\mu(u+\omega_1+\omega_2)^2. \tag{4.28}$$

If (4.28) accomplishes, (4.27) and (4.28) give the relations that for $v := u - \frac{1}{2}\omega_2$,

$$-2\wp_{22}(v \pm \tfrac{1}{2}\omega_2) - \lambda_4 - a_2 = \mu^2(v+\omega_1+\omega_2) \pm \partial_{u_2}\mu(v+\omega_1+\omega_2). \tag{4.29}$$

Using (H1) in Proposition 4.5, we can show that $q := -2\wp_{22}(v \pm \frac{1}{2}\omega_2) - \lambda_4 - a_2$ obeys

$$-4\partial_{u_1}q + 6q\partial_{u_2}q + \partial_{u_2}^3 q = 0 \tag{4.30}$$

if $a_2 = -\frac{1}{3}\lambda_4$. From Proposition 2.2, if (4.28) is true, Lemma 4.8 will be proved.

Thus we will concentrate (4.28). We add (4.27) to $2\wp(u) + \lambda_4 + a_2$ and then we have the relation

$$\wp_{22}(u+\omega_2) + \wp_{22}(u) + \lambda_4 + a_2 = -\mu(u+\omega_1+\omega_2)^2 + \frac{\Delta(u+\omega_1+\omega_2)}{4\rho(u+\omega_1+\omega_2)^2}, \tag{4.31}$$

where

$$\Delta(u) := 8\wp_{22}(u)\rho(u)^2 - 2(\partial_{u_2}^2\rho(u))\rho(u) + (4\lambda_4 + a_2 + 1)(\partial_{u_2}\rho_2(u))^2. \tag{4.32}$$

We use the relations in Proposition 4.3 and then direct computation shows that Δ vanishes even though it is very tedious. Accordingly the lemma is proved. □

Theorem 4.4 is also proved and Remark 3.6 is also effective in this case.

5. Discussion

We showed that the loop soliton is directly connected with σ-functions. In Ref. [11], I partially showed the fact but I could not reach the concrete function form and the difference between numerator and denominator in the ratio of σ- or τ-function. This study determines that the difference comes from half of the period or the so-called theta characteristics [2,10]. Even though we could not find a solution with higher genus, it is not difficult to conjecture the function form of loop soliton in terms of the theta characteristics.

As we used the definition of loop soliton as a relation of MKdV equation, our study is just that of MKdV equation. In other words, we have concretely constructed the function forms of the genus one and the genus two periodic solutions of MKdV equation in Sections 3 and 4. In ordinary study of elliptic function solution of MKdV equation [11], one uses Jacobi elliptic functions, sn, cn, and dn functions. Our solution (3.15) becomes the Jacobi elliptic function by means of Landen transformation, Jacobi transformation, Gauss transformation and so on. The relation between Weierstrass and Jacobi elliptic functions is derived from (3.14) [21]. $\sqrt{\wp(u) - e_a}$ plays very important role:

$$\mathrm{sn}(z) = \sqrt{\frac{e_1 - e_3}{\wp(u) - e_3}}, \quad \mathrm{cn}(z) = \sqrt{\frac{\wp(u) - e_1}{\wp(u) - e_3}}, \quad \mathrm{dn}(z) = \sqrt{\frac{\wp(u) - e_2}{\wp(u) - e_3}}, \tag{5.1}$$

where $z = (e_1 - e_3)^{1/2}u$. Since sn function is connected with 1/sn by phase shift and associated with dn and cn functions by algebraic equations, the Jacobi elliptic function could roughly be considered as $\sqrt{\wp(u) - e_3}$. Similarly, we might regard $\sqrt{\rho}$ in (4.25) as a Jacobi-type hyperelliptic function due to (4.20); $\sqrt{\rho/a_2}$ is essentially $\sqrt{\wp(u) - e_3}$ when we set $\partial_{u_1}\sigma = 0$ and $u := u_2$.

We should also notice the square in (3.14) and (4.20). As described in [11] and reference therein, it is closely related to fermion. In fact, $\psi := \sigma_2/\sigma \sim \sqrt{\partial_s Z}$ in (3.14) ($\psi := \sigma_2/\sigma_0$ in (4.20)) obeys the Dirac equation

$$\begin{pmatrix} \partial_s & \mu \\ \mu & -\partial_s \end{pmatrix} \begin{pmatrix} \psi \\ \sqrt{-1}\psi \end{pmatrix} = 0. \tag{5.2}$$

The square root of $\partial_s Z$ means double covering of the tangent bundle. (5.2) is related to Schwarz derivative and moduli space as mentioned in [11]. We recognize that there is still an open problem, what is the origin of the square root.

Finally, we will comment upon the closed condition mentioned in Remark 3.6. We could not touch the condition. In general, an integration of periodic function is not periodic. However if the function satisfies some conditions, its integral is also periodic. The closed condition is such a condition, it suppresses two degrees of freedom in the parameter spaces. For the study of quantized elastica problem, it is very important to determine the relation between the conditions and the coefficients of the algebraic curve λ_j in (4.1). In future, we wish to determine them.

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